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Linear astral (n_5) configurations with dihedral symmetry

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Abstract

A linear astral (n_k) configuration is a collection of points and straight lines, so that each point lies on k lines and each line passes through k points, with $\lfloor \frac{k+1}{2} \rfloor$ symmetry (transitivity) classes of points and lines under rotations and reflections mapping the configuration to itself. We discuss the possible structures of astral (n_5) configurations with dihedral symmetry group D_m in the Euclidean plane, and we provide methods to investigate the existence of such configurations. In doing so, we introduce a new class of astral (n_3) configurations.

In the present paper, an (n_k) configuration is a collection of n points and n straight lines in the Euclidean plane, such that each point has k lines passing through it and each line has five points lying on it. An (n_k) configuration is *astral* if it has $\lfloor \frac{k+1}{2} \rfloor$ symmetry classes (orbits) of points and $\lfloor \frac{k+1}{2} \rfloor$ symmetry classes of lines under configuration-preserving rotations and reflections of the plane. In this paper, we will focus our attention on (n_5) configurations with dihedral symmetry group D_m, where m is such that the configuration is astral. Although other astral configurations have been studied [2,3,6–8], with the exception of Grünbaum [6, p. 220] who provides examples of (n_5) configurations that are astral in the projective (extended Euclidean) plane, the case of astral (n_5) configurations has not yet been studied.

The following conjecture was stated by Grünbaum in [6, p. 220] and by the first author in [2, p. 28]:

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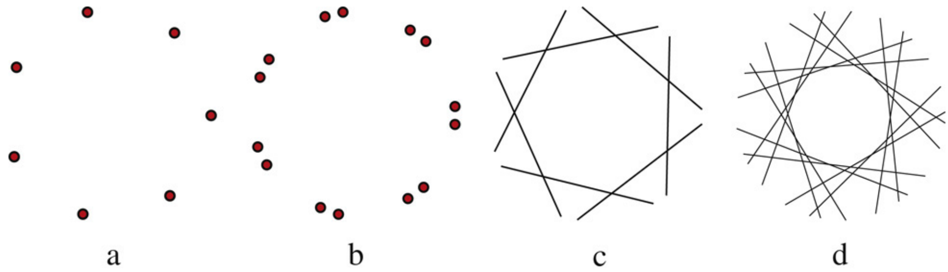

Fig. 1. Structures for point and line classes, geometrically. (a) $\mathcal{A}_1$; (b) $\mathcal{A}_2$; (c) $\mathcal{L}_1$; (d) $\mathcal{L}_2$. Here, $m = 7$.

Conjecture 1. *No linear astral* (n_5) *configurations exist in the Euclidean plane.*

In this paper, we show that there are only two categories of astral (n_5) configurations with dihedral symmetry. We provide confirmation of Conjecture 1 for $n < 3000$ for one category of linear astral (n_5) configurations, and in doing so, we develop a method of constructing a previously unknown class of astral (n_3) configurations. Finally, we provide an approach for investigating a second category of configurations, and show that there are no (n_5) configurations of this category for $n < 120$.

While in this paper we focus on astral (n_5) configurations with dihedral symmetry, there are other interesting open questions regarding (n_5) configurations. For astral (n_5) configurations, these include classifying astral (n_5) configurations with cyclic symmetry (it is likely that these do not exist, either, but the tools developed in this paper seem inapplicable) and classifying astral (n_5) configurations in the projective plane. There are more general questions as well: what can we say about (n_5) configurations with non-trivial geometric symmetry? At present, only a few examples of symmetric (n_5) configurations are known, and in general, symmetric (n_k) configurations for $k > 4$ are poorly understood.

1. Structure of dihedrally symmetric astral (n_5) configurations

We study astral (n_5) configurations with dihedral symmetry group D_m; that is, each configuration has the symmetries of a regular m-gon for some m. Suppose such a configuration exists. For convenience, assume the regular m-gon is centered at the origin. For each class of points, we have two possibilities (examples are shown in Fig. 1(a) and (b)):

$\mathcal{A}_1$ The symmetry class of points forms a regular m-gon;
$\mathcal{A}_2$ The symmetry class of points forms an isogonal $2m$-gon; that is, $2m$ points are distributed on a circle forming two concentric regular m-gons.

Since there are three symmetry classes of points, *a priori*, there are the following (unordered) possibilities for the types of symmetry classes of points, shown in the left-hand column of Table 1.

Now consider the three symmetry classes of lines. With each class of lines, we can associate its polar dual, a set of points, using as the circle of inversion some circle centered at the origin. Therefore, the above structure for point classes carries over to line classes: for each set of lines, the dual set of points must satisfy either property $\mathcal{A}_1$ or property $\mathcal{A}_2$. Examples of the line structure are shown in Fig. 1(c) and (d). Hence,

$\mathcal{L}_1$ The polar dual of the symmetry class of lines forms a regular m-gon;
$\mathcal{L}_2$ The polar dual of the symmetry class of lines forms an isogonal $2m$-gon.

Table 1
Possible structures for points and lines in an astral (n_5) configuration

Points	n	Lines	n
$(\mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_1)$	$3m$	$(\mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_1)$	$3m$
$(\mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_2)$	$4m$	$(\mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_2)$	$4m$
$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_2)$	$5m$	$(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_2)$	$5m$
$(\mathcal{A}_2, \mathcal{A}_2, \mathcal{A}_2)$	$6m$	$(\mathcal{L}_2, \mathcal{L}_2, \mathcal{L}_2)$	$6m$

As with the points, there are four possibilities for the structure of all three line classes, listed in the right-hand column of Table 1. Since the number of points and the number of lines in an (n_5) configuration are equal, only the following combinations need to be considered:

Case 1: $(\mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_1)$ and $(\mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_1)$, with $n = 3m$;
Case 2: $(\mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_2)$ and $(\mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_2)$, with $n = 4m$.
Case 3: $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_2)$ and $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_2)$, with $n = 5m$;
Case 4: $(\mathcal{L}_2, \mathcal{L}_2, \mathcal{L}_2)$ and $(\mathcal{A}_2, \mathcal{A}_2, \mathcal{A}_2)$, with $n = 6m$.

However, not all of these cases are possible. If a symmetry class of lines has property $\mathcal{L}_1$, then its polar points form a regular m-gon, so each line is perpendicular to a line passing through the origin which is a mirror of the m-gon. If, in addition, a symmetry class of points has property $\mathcal{A}_1$, then each point in that class lies on one of the mirrors of the m-gon. Therefore, if an astral (n_5) configuration possesses a class of lines L with property $\mathcal{L}_1$ and a class of points P with property $\mathcal{A}_1$, since each line must contain at least one point from each symmetry class of points, the lines L are tangent to the circumcircle of the points P at those points.

One consequence is that neither Case 1 nor Case 2 is possible, since we can assume that the classes of points lie at different distances from the origin. If there were two sets of lines L_1 and L_2 with property $\mathcal{L}_1$, the representatives incident with points with property $\mathcal{A}_1$ are both tangent to the circumcircles of those points and therefore must coincide.

Theorem 2. *There are only two possibilities for the structure of an astral (n_5) configuration with dihedral symmetry: $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_2)$ and $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_2)$, with $n = 5m$ points, and $(\mathcal{A}_2, \mathcal{A}_2, \mathcal{A}_2)$ and $(\mathcal{L}_2, \mathcal{L}_2, \mathcal{L}_2)$ with $n = 6m$ points.*

An astral (n_5) configuration with structure $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_2)$ and $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_2)$ is called a *category* 1 configuration and an astral (n_5) configurations with structure $(\mathcal{A}_2, \mathcal{A}_2, \mathcal{A}_2)$ and $(\mathcal{L}_2, \mathcal{L}_2, \mathcal{L}_2)$ is called a *category* 2 configuration.

2. Category 1 astral (n_5) configurations, with $5m$ points

A category 1 astral (n_5) configuration has a single class of lines L_0 whose polar points form a regular m-gon and a single class of points P_0 that form a regular m-gon; these are called the *special class* of lines and points, respectively.

Lemma 3. *In a category 1 astral (n_5) configuration, every line passes through one point from the special class of points and two points from each of the other two classes. Likewise, each point lies on one line from the special class and two lines from each of the other two classes.*

Proof. In a category 1 $((5m)_4)$ configuration, suppose the special class of points and lines are labelled P_0 and L_0 respectively, and the other two classes of points and lines are called P_1, P_2,

L_1 and L_2. Note that the special class of points P_0 participates in $5m$ point–line incidences (5 lines/point $\times$ the number of points).

Suppose two points from P_0 lie on some line ℓ. By symmetry, two points from P_0 must lie on every element of $D_m(\ell)$ of ℓ, which is one of the three symmetry classes. At least one point from P_0 must lie on each of the other two symmetry classes, since a maximum of two points from a single symmetry class may lie on a line. Without loss of generality, if $D_m(\ell) = L_1$ then there are $2 \cdot 2m$ point–line incidences from P_0 and L_1 and at least $2m$ incidences from L_2 and m from L_0, for a total of at least $7m$ point–line incidences. If $D_m(\ell) = L_0$, there are $2 \cdot m$ incidences involving P_0 and L_0 and at least $2m$ incidences from each of the other two classes, for a total of at least $6m$ point–line incidences involving P_0. Either way, there are too many point–line incidences involving P_0. $\square$

An astral (n_3) configuration has three points per line, three lines per point, and two symmetry classes each of points and lines. If a category 1 astral $((5m)_5)$ configuration exists, then by ignoring one of the non-special classes of points and lines, four underlying astral (n_3) configurations are formed with symmetry group D_m and structure $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{L}_1, \mathcal{L}_2)$. In particular, if the $((5m)_5)$ configuration has point classes P_0, P_1, P_2 where P_0 is special and line classes L_0, L_1, L_2 where L_0 is special, the underlying astral (n_3) configurations each contain P_0 and L_0 and one of the other two point and line classes, so that the possible configurations have point and line classes $(L_0, L_1; P_0, P_1)$, $(L_0, L_1; P_0, P_2)$, $(L_0, L_2; P_0, P_1)$ or $(L_0, L_2; P_0, P_2)$. To determine the existence of astral (n_5) configurations, we can ask whether we can find pairs of astral (n_3) configurations of the form $(L_0, L_1; P_0, P_1)$ and $(L_0, L_2; P_0, P_2)$; combining such pairs accounts for all of the symmetry classes from the (n_5) configuration.

To construct dihedrally symmetric astral (n_3) configurations in the Euclidean plane of structure $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{L}_1, \mathcal{L}_2)$, we do the following. Construct a regular m-gon centered at the origin $\mathcal{O}$ with vertices numbered $0, 1, \ldots, m-1$, where

$$\text{vertex } i = \left(\cos\left(\frac{2\pi i}{m}\right),\ \sin\left(\frac{2\pi i}{m}\right)\right),$$

and the vertical line t passing through vertex $0 = (1, 0)$. Define $C_m(i, j)$ as the circle that passes through vertices i and j and the origin, where entries i and j are taken modulo m. Construct $\mathcal{C} = C_m(a, -a)$, where a is chosen so that $\mathcal{C}$ intersects t in two places, say X above and Y below the vertex $0 = (1, 0)$. Let $Z_m(Q)$ be the orbit of an object Q (either points or lines) under rotation about the origin by integer multiples of $\frac{2\pi}{m}$. Then the points of the astral (n_3) configuration are $Z_m(0)$, $Z_m(X)$ and $Z_m(Y)$, and the lines of the configuration are $Z_m(t)$, $L_X = Z_m(\langle X, a\rangle)$ and $L_Y = Z_m(\langle Y, -a\rangle)$, where $\langle X, a\rangle$ is the line connecting X and a. See Fig. 2 for an example of this construction, for $m = 14$ and $a = 3$.

Theorem 4. *The above construction produces an astral (n_3) configuration.*

Note that these configurations form a new class of astral (n_3) configurations; they were not discussed in [6].

To prove Theorem 4, we need the following two lemmas. Lemma 5 is a well-known theorem from Euclidean geometry.

Lemma 5. *If AB and $A'B'$ are equal chords of a circle $\mathcal{C}$ and P and Q are points on $\mathcal{C}$, then $\angle APB = \angle A'QB'$.*

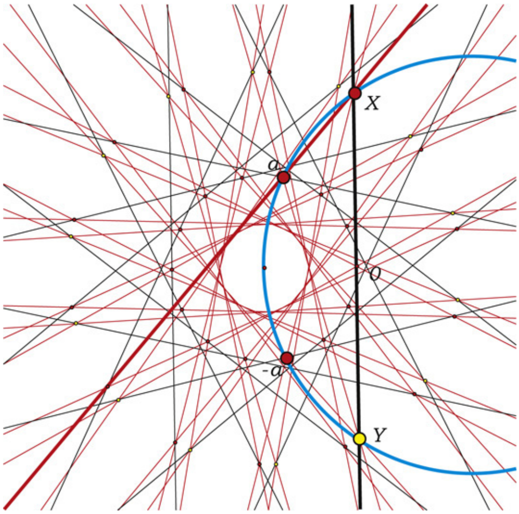

Fig. 2. Construction of astral (n_3) configurations; here $m = 14$ and $a = 3$.

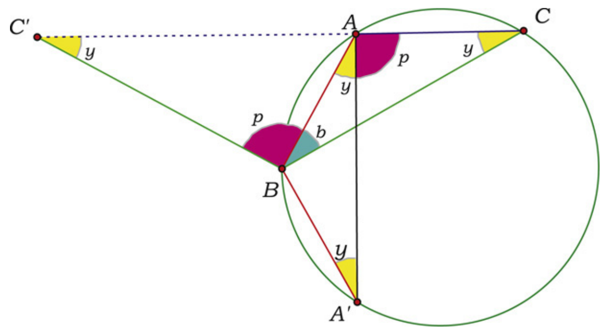

Fig. 3. Diagram for the proof of Lemma 6.

Lemma 6. *Suppose A, B, A' and C are points on a circle, where $AB = BA'$. If C is rotated about B so that $\angle CBC' = \angle A'BA$, then the points C, A and C' are collinear.*

Proof. Let angle $BAA' = y$ (see Fig. 3). By hypothesis, $AB = BA'$, so by Lemma 5, $\angle ACB = y$, since it is also subtended by chord BA. Let $\angle ABC = b$ and $\angle CAA' = p$ and note that $\angle BAC = y + p$. From $\triangle ABC$, we conclude $2y + p + b = 180$. By construction, $\angle ABA' = \angle CBC'$ and $\triangle ABA'$ and $\triangle CBC'$ are both isosceles, so $\triangle ABA' \sim \triangle CBC'$ and hence $\angle CC'B = y$. Therefore, $\angle ABC' = 180 - 2y - b = p$ (whether or not A lies on CC'), so in $\triangle C'AB$, $\angle C'AB = 180 - y - p = y + b$. Finally, note that $\angle C'AC = C'AB + BAC = (y + b) + (y + p) = 180$, so the points C, A and C' lie on a straight line. □

Using this lemma, we prove Theorem 4.

Proof of Theorem 4. To show this construction produces an astral (n_3) configuration, we need to show that each point is incident with three lines, each line contains 3 points, and that there are two symmetry classes of points and lines. Note that the vertices labelled a and $-a$ (with labels taken modulo m) are images of each other under reflection over the x-axis, so the circle $\mathcal{C}$ has

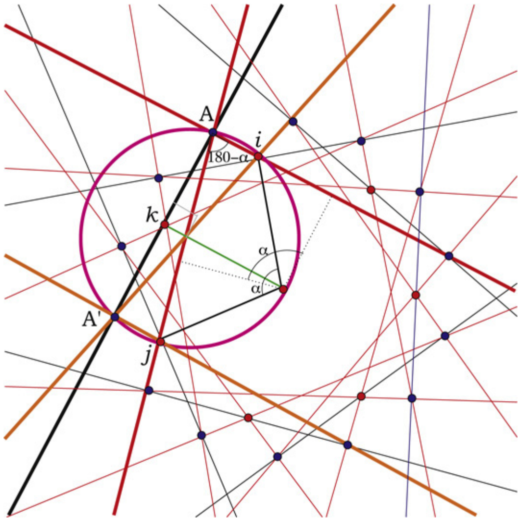

Fig. 4. Diagram used in the proof of Theorem 7. Points in $Z_m(i)$ are red, points in $D_m(A)$ are blue, lines in $Z_m(\ell)$ are red and lines in $Z_m(t)$ are black. Lines ℓ_i and ℓ_j and their reflections over $\mathcal{O}k$ are thick. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the x-axis as a mirror. Consequently, X and Y, and also $\langle X, a\rangle$ and $\langle Y, -a\rangle$, are mirror images of each other. Therefore, the orbit of X is $Z_m(X) \cup Z_m(Y)$ and the orbit of $\langle X, a\rangle$ is $L_X \cup L_Y$. It remains to show that each line in the orbit of $\langle X, a\rangle$ contains three points. We will show $\langle X, a\rangle$ contains two points from $Z_m(X)$ along with one point from $Z_m(0)$. Note that if X_i is the image of X under rotation by $\frac{2\pi i}{m}$ about $\mathcal{O}$, then the angle between $X = X_0$ and X_{2a} is equal to the angle between vertex a and $-a$. By construction, a, $-a$, X and $\mathcal{O}$ all lie on a circle. Therefore, by applying Lemma 6, using $B = \mathcal{O}$, $A = a$, $A' = -a$, $C = X$ and $C' = X_{2a}$ it follows that the points X, a and X_{2a} are all collinear. $\square$

In fact, we can say something stronger.

Theorem 7. *Every astral (n_3) configuration with dihedral symmetry with structure $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{L}_1, \mathcal{L}_2)$ may be constructed as described above.*

To see this, we need the following lemma from Euclidean geometry:

Lemma 8 (*Cyclic Quadrilaterials*). *A quadrilateral $PQRS$ may be inscribed in a circle if and only if opposite angles are supplementary.*

Proof of Theorem 7. Suppose that an (n_3) configuration $\mathcal{C}$ has structure $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{L}_1, \mathcal{L}_2)$; this implies the special class of lines is tangent to the circumcircle of the special class of points at those points. Suppose the special classes of points and lines are called $Z_m(i)$ and $Z_m(t)$, respectively, and the non-special classes of points and lines are called $D_m(A)$ and $D_m(\ell)$, where $D_m(Q)$ is the orbit of an object Q under the dihedral group of order m. Note that the entire configuration has the dihedral symmetries of an m-gon and that all lines of the form $\mathcal{O}i$, for i in $Z_m(i)$ are mirrors of the configuration. Suppose that $\mathcal{C}$ is positioned so that its center is at the origin $\mathcal{O}$ and one of the special class of lines, called t, is perpendicular to the x-axis. The point A in $D_m(A)$ (see Fig. 4) has three lines passing through it, one from $Z_m(t)$, called t' and two from

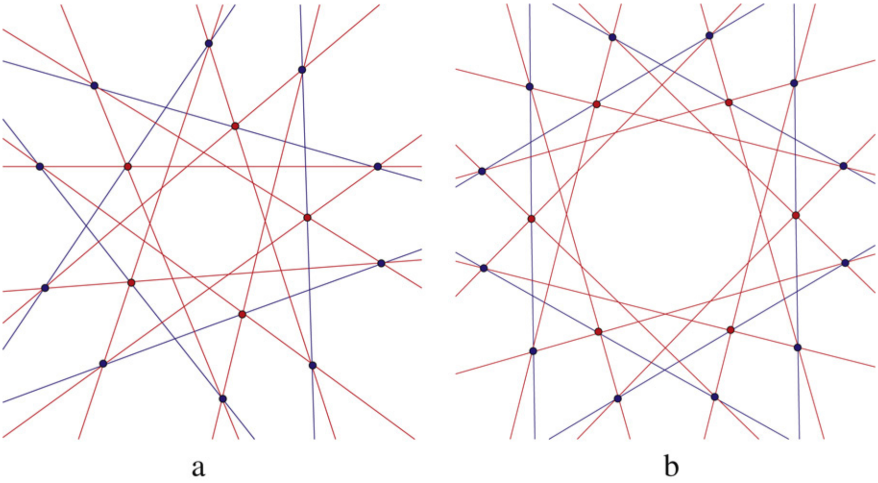

Fig. 5. The two smallest astral (n_3) configurations whose special class of lines is tangent to the circumcircle of the special class of points at those points. (a) A (15_3) configuration ($m = 5$, $a = 1$); (b) an (18_3) configuration ($m = 6$, $a = 1$).

$D_m(\ell)$. Because each line contains two points from $D_m(A)$ and one from $Z_m(i)$, t' contains a point k from $Z_m(i)$ and another point A' from $D_m(A)$. By hypothesis, since the special class of lines is tangent to the circumcircle of the special class of points at those points, A and A' must be mirror images under reflection through line $\overline{\mathcal{O}k}$.

Now consider the two lines from $D_m(\ell)$ passing through A. These each contains one point of $Z_m(i)$, say points i and j, and we label the two lines as ℓ_i and ℓ_j. If $\angle i\mathcal{O}j = \alpha$, then ℓ_j is the rotated image of ℓ_i through rotation by α. Since the special class of points must be closer to the center of the configuration than the non-special class, angle $\angle iAj = 180 - \alpha$, since segment ij separates A and $\mathcal{O}$. That is, angles $\angle iAj$ and $\angle i\mathcal{O}j$ are supplementary, so points i, j, A and $\mathcal{O}$ are concyclic.

Since line $\overline{\mathcal{O}k}$ is a mirror for the configuration and A and A' are reflected images, the lines in $D_m(\ell)$ that intersect A' are the reflections of ℓ_i and ℓ_j and also contain i and j, respectively. Using the same argument as before, i, j, $\mathcal{O}$ and A' are also concyclic, so all five points i, j, A, A' and $\mathcal{O}$ lie on the same circle and i and j are reflections of each other. Therefore, the number of points between i and k in $Z_m(i)$ is equal to the number of points between k and j; in particular, this means that i and j are of the same parity. To construct the original (n_3) configuration, it would suffice to use the line t that is perpendicular to the x-axis and the circle $C_m(a, -a)$ where $a = \frac{i-j}{2}$. □

The two smallest astral (n_3) configurations constructed using this method are shown in Fig. 5, for $m = 5$ and $m = 6$.

Theorem 9. *There are no category* 1 *astral* (n_5) *configurations for* $n < 3000$.

To construct category 1 astral (n_5) configurations, we need to be able to find two circles $C_m(i, m-i)$ and $C_m(j, m-j)$ with $i \neq j$, which have the property that $C_m(i, m-i)$ and some rotated image of $C_m(j, m-j)$ both intersect the tangent line t in two places, and one of the points of intersection is the same (that is, the two circles intersect each other on t). Note that if $C_m(j, m-j)$ is rotated by k steps, the resulting circle is $C_m(k+j, k-j)$ (with indices taken

Table 2
Executable code used to determine whether there exist compatible circles, written in the functional programming language Haskell

```
is::Integer->[Integer]; is m|mod m 4 == 0 = [1..(div m 4)-1]|otherwise=[1..div m 4]

js::Integer->Integer->[Integer]
js m i |mod m 4 == 0 = [1..(div m 4)-1]\\[i]|otherwise =[1..div m 4]\\[i]

midpt1::Integer->Integer->(Double,Double)
midpt1 m i = (1/(2*cos((fromInteger i)*2*pi/(fromInteger m))),0)

ks::Integer->Integer->[Integer]
ks m j = [1..kmax]++map negate [1..kmax]
  where rj::Double;rj = 1/(2*cos((fromInteger j)*2*pi/(fromInteger m)))
      kmax=toInteger(fromEnum(((fromInteger m)*(atan((sqrt(2*rj-1))/(1-rj))))/(2*pi)))

midpt2::Integer->Integer->Integer->(Double,Double)
midpt2 m j k = (rj,(fromInteger k)*2*pi/(fromInteger m))
  where rj = 1/(2*cos((fromInteger j)*2*pi/(fromInteger m)))

ht1::(Double,Double)->Double; ht1 (r,phi) = r*(sin phi)+sqrt(r^2-(1-r*cos phi)^2)

ht2::(Double,Double)->Double; ht2 (r,phi) = r*(sin phi)-sqrt(r^2-(1-r*cos phi)^2)

dist1::Integer->Integer->Integer->Integer->Double
dist1 m i j k = abs((ht1(midpt1 m i))-(ht1(midpt2 m j k)))

dist2::Integer->Integer->Integer->Integer->Double
dist2 m i j k = abs((ht1(midpt1 m i))-(ht2(midpt2 m j k)))

pm::Integer; pm = 100

er::Double; er = 1.0e-9

tuples::[[Integer]]
tuples=[[m,i,j,k]|p<-[1..pm], m<-[6*p], i<-is m, j<-js m i, k<-ks m j,
        [m,i,j,k]`notElem`[[24*l,4*l,3*l,3*l]|l<-[1..div pm 4]],
      dist1 m i j k<er || dist2 m i j k<er ]
```

modulo m); thus, the constraint is that given i and j, we can find some k so that $C_m(i, m-i)$ and $C_m(k+j, k-j)$ intersect t at the same point. We say such circles are *compatible*. The existence of two compatible circles is a necessary condition for the construction of a category 1 astral (n_5) configuration; it is also sufficient, under the assumption that the intersection point of the two circles does not lie on a line of symmetry of the m-gon.

To determine the existence of compatible circles, we first determine values of i and j for which circles $C_m(i, m-i)$ and $C_m(j, m-j)$ intersect the tangent line t twice; in particular we consider $1 \leq i, j < \frac{m}{4}$ with $i \neq j$. Functions `is` and `js` in the computer code given in Table 2, using the functional programming language Haskell, provide the corresponding lists of indices. (See http://www.haskell.org/ for more information on the programming language Haskell, in particular [10]). Next, we determine for a fixed j a value $k_{\max}$ so that for all k satisfying $1 \leq k \leq k_{\max}$, circles $C_m(k+j, k-j)$ intersect t twice; i.e., we choose those values of k so that the center of $C_m(k+j, k-j)$ lies to the right of the parabola $2x = 1 - y^2$. The function `midpt1` returns the center of the circle $C_m(i, m-i)$ in polar coordinates. For each relevant quadruple $[m, i, j, k]$ we compute the intersection points with the tangent t of circles $C_m(i, m-i)$ and $C_m(k+j, k-j)$ using the functions `ht1` and `ht2` and compute the distances between the intersection points, using `dist1` and `dist2`. If the distance is greater than our chosen error `er=1.0e(-9)`, then those two circles are not compatible. It is necessary to eliminate consideration of circles that intersect the tangent line on a line of symmetry of the

m-gon, since in that case, the incidence structure that is produced by the circles is not a configuration (points in the special class lie on only three lines). For example, circles $C_{24}(4, 20)$ and $C_{24}(6, 0)$ intersect the line $x = 1$ at the point $y = 1$, but the line $y = x$ is a line of symmetry for the 24-gon, since it passes through vertex 3, so the quadruple $[m, i, j, k] = [24, 4, 3, 3]$ is eliminated. In general, if $[m, i, j, k]$ is a quadruple that does not correspond to compatible circles and so may be discarded, all quadruples $[mq, iq, jq, kq]$ for integers $q \geq 1$ also may be discarded, so in particular, we discard all quadruples $[24q, 4q, 3q, 3q]$. The function `tuples` provides a list of quadruples that has discarded the above quadruples. Using this method, we investigated whether we can find compatible circles by performing numerical computations using the code given in Table 2. For $n < 3000$, the output of `tuples` was an empty list, meaning that there are no pairs of compatible circles and hence no category 1 (n_5) configurations.

3. Category 2 astral (n_5) configurations, with $6m$ points

A category 2 astral (n_5) configuration has three classes of points which all form isogonal m-gons and three classes of lines whose polar points also form isogonal m-gons.

Lemma 10. *If we label the line classes of a category* 2 *astral* $((6m)_5)$ *configuration as* L_0*,* L_1 *and* L_2 *and the point classes as* P_0*,* P_1 *and* P_2*, then every line in class* L_i *is incident with two points from class* P_{i+1} *and* P_{i+2} *and one point from class* P_i*, with indices taken modulo* 3*. Likewise, every point in class* P_i *has two lines from class* L_{i+1} *and* L_{i+2} *passing through it and one line from class* L_i*.*

Proof. Suppose not. That is, without loss of generality, suppose that each line in class L_0 contains two points from P_1 and P_2 and one point from P_0 and that each line in class L_1 also contains two points from P_1 and P_2 and one point from P_0. Each line in L_2 contains at most two points from P_0.

In a $((6m)_5)$ configuration, each class of points contains $2m$ points, and each point in the class has five lines passing through it. Therefore, there should be $5 \cdot (2m) = 10m$ point–line incidences involving each class of points (and all line classes). If we consider points in class P_0, there are $2m$ point–line incidences from lines in L_0 (since they each contains one point in P_0 and there are $2m$ lines), $2m$ point–line incidences from lines in L_1, and at most $2 \cdot 2m$ point–line incidences from lines in L_2, for a total of at most $8m$ point–line incidences involving points in P_0; this contradicts the necessary number of point–line incidences for P_0. □

To determine if there exist any category 2 astral $((6m)_5)$ configurations, we again can attempt to find pairs of configurations which, when combined, form the $((6m)_5)$ configuration.

Suppose we begin with a category 2 astral $((6m)_5)$ configuration with the symmetries of the regular m-gon $\mathcal{M}$ whose vertices are $(\cos(\frac{2\pi i}{m}),\ \sin(\frac{2\pi i}{m}))$. Suppose its point classes are called P_0, P_1, P_2 and its line classes are L_0, L_1, L_2. Note that none of the points can lie on any of the mirrors of the m-gon, since each point class must form an isogonal $2m$-gon whose symmetries are that of $\mathcal{M}$. If S is a set of objects, let S' refer to the set of objects formed by reflecting each object in S over the x-axis, and if t is a particular object, let $Z_m(t)$ refer to the images of t under rotation by integer multiples of $\frac{2\pi}{m}$ about the origin. Choose a point p_0 in point class P_0 and note that P_0 is partitioned into $Z_m(p_0)$ and $Z_m(p_0)'$. By hypothesis, p_0 is incident with two lines from L_1. Call one of the two lines l_1. Then $Z_m(l_1)$ and $Z_m(l_1)'$ partition L_1, and because of the symmetry constraints inherited from the original configuration, both lines from L_1 that pass through p_0 must be elements of $Z_m(l_1)$.

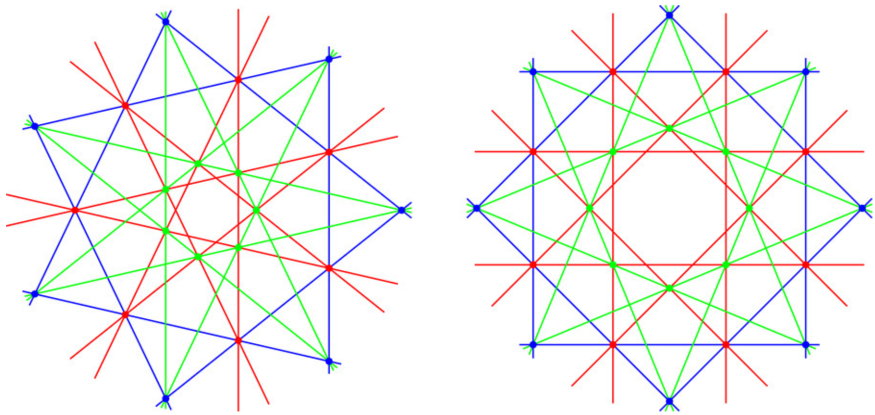

Fig. 6. Celestial $((3m)_4)$ configurations for $m = 7, 8$ with symbols $7\#(2,1;3,2;1,3)$ and $8\#(2,1;3,2;1,3)$ respectively.

Points in set P_2 also contain two lines from L_1. Suppose that p_2 is a point in P_2 that lies on the intersection of two lines in $Z_m(l_1)$; note that because P_2 forms an isogonal $2m$-gon, its points cannot lie on the intersection of one line from $Z_m(l_1)$ and one line from $Z_m(l_1)'$ (because in that case they would form a regular m-gon). Then P_2 is partitioned into $Z_m(p_2)$ and $Z_m(p_2)'$.

Now, points in $Z_m(p_2)$ are incident with two lines in L_1, discussed previously, and two lines in L_0. If one of the lines from L_0 is called l_0, then $Z_m(l_0)$ and $Z_m(l_0)'$ partition L_0 and all points in $Z_m(p_2)$ lie on lines in $Z_m(l_0)$. The lines L_0 are also incident with points in P_1. If one of these points is labelled p_1, then $Z_m(p_1)$ and $Z_m(p_1)'$ partition P_1, and lines in $Z_m(l_0)$ only pass through points in $Z_m(p_1)$.

Since points in P_1 also have lines from L_2 passing through them, there is a line l_2 in L_2 that passes through p_1, and again, L_2 will be partitioned into $Z_m(l_2)$ and $Z_m(l_2)'$. Finally, since points in P_0 also lie on lines of L_2 and because of the assumptions that have already been made, the points in $Z_m(p_0)$ must lie on lines in $Z_m(l_2)$. The sequence of incidences is therefore $P_0 \to L_1 \to P_2 \to L_0 \to P_1 \to L_2 \to P_0$.

Moreover, since each line must pass through five points, a line from $Z_m(l_i)$ must pass through one point in $Z_m(p_i)'$, because if a line in $Z_m(l_i)$ were to pass through a point in $Z_m(p_i)$, the symmetries of the configuration would force it to pass through two points, which is forbidden.

If we just consider the points in $Z_m(p_0)$, $Z_m(p_1)$ and $Z_m(p_2)$ and the lines in $Z_m(l_0)$, $Z_m(l_1)$ and $Z_m(l_2)$, the above discussion shows that each point in $Z_m(p_i)$ lies on two lines of $Z_m(p_{i+1})$ and two lines of $Z_m(p_{i+2})$ and likewise, each line in $Z_m(l_i)$ contains two points of $Z_m(p_{i+1})$ and two points of $Z_m(p_{i+2})$. That is, the collection of these points and lines forms a $((3m)_4)$ configuration. In fact, these configurations are known; they are 3-ring *celestial* (n_4) configurations (see [1] and [6, p. 201–3] for details). Examples of 3-ring celestial configurations for $m = 7, 8$ are shown in Fig. 6. Following the most recent notation for celestial 3-ring configurations, given in [1] (a modification of the notation introduced in [6]) every 3-ring celestial configuration may be represented by a configuration symbol $m\#(s_1, t_1; s_2, t_2; s_3, t_3)$, where m is the number of points in each ring. A complete description of how to construct celestial configurations is given in [1].

To determine the existence of category 2 astral $((6m)_5)$ configurations, we construct celestial $((3m)_4)$ configurations that potentially may participate in a Category 2 $((6m)_5)$ configuration. We require that $t_1 < s_1$ and $t_2 < s_2$ so that the first ring of points as designated by the configuration

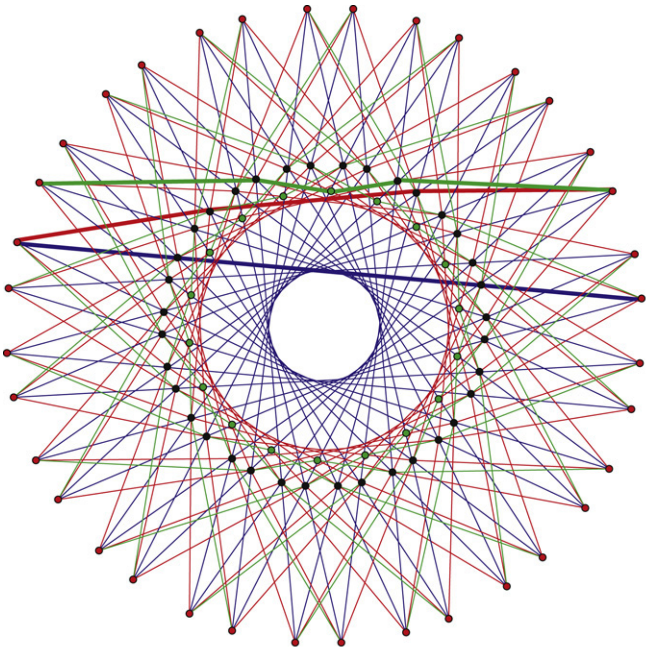

Fig. 7. A category 1 astral (90_5) configuration of pseudolines, with $m = 18$.

symbol is actually the outermost ring of points. In addition, we only consider configurations where both classes of lines incident with the outermost circle of points intersect both circles passing through the inner symmetry classes of points. We choose one representative for each polar pair of configurations. Given a potential configuration, if there exists a line through the origin which could serve as a line of symmetry of the $((6m)_5)$ configuration, so that if one point from each point class of the configuration is reflected over this line, each reflected point lies on a line of the original configuration, then the configuration would form an astral (n_5) configuration by reflecting the entire configuration over that line. We have verified that none of the potential celestial configurations combine in this way to form a category 2 astral (n_5) configuration for $m \leq 20$. That is:

Theorem 11. *There are no category* 2 *astral* (n_5) *configurations for* $n \leq 120$.

4. Astral (n_5) configurations of pseudolines

While no examples of linear astral (n_5) configurations are known in the Euclidean plane, even without the assumption of dihedral symmetry, we can provide dihedrally symmetric astral (n_5) pseudoline configurations. Since pseudolines, equivalent to rank 3 oriented matroids, are not our main topic, we refer the reader for possible definitions to Grünbaum, *Arrangements and Spreads* [9], Björner et al., *Oriented Matroids* [5] or Bokowski, *Computational Oriented Matroids* [4]. Here, we simply note that we can provide examples for both category 1 and category 2 astral (n_5) configurations of pseudolines with dihedral symmetry, shown in Figs. 7 and 8.

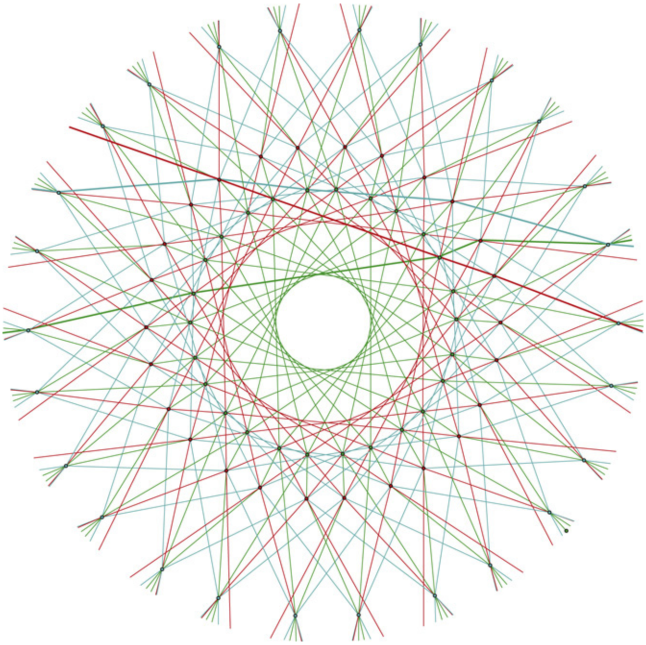

Fig. 8. A category 2 astral (78_5) configuration of pseudolines, with $m = 13$.

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